Supplemental Document

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Generalized Wigner–Smith theory for perturbations at resonant exceptional and diabolic point degeneracies: supplement

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1. REVIEW OF LINEAR ALGEBRA CONCEPTS

Here we review concepts from linear algebra that are most pertinent to the main text, focusing on eigenvalues, algebraic and geometric multiplicities, and the Jordan normal form. A more in-depth treatment can be found, for example, in Ref. [1].

The eigenvalues of a matrix $\mathbf{M}$ are the roots of its characteristic equation $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$. If all the eigenvalues of $\mathbf{M}$ are distinct, then $\mathbf{M}$ is diagonalizable: there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is diagonal with the eigenvalues of $\mathbf{M}$ on the diagonal. The columns of $\mathbf{P}$ are the corresponding eigenvectors, which can be found by solving $(\mathbf{M} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$ for each eigenvalue λ.

If the characteristic equation has repeated roots, $\mathbf{M}$ may or may not be diagonalizable. The number of times an eigenvalue occurs as a root of the characteristic polynomial is called its *algebraic multiplicity* (AM). The number of linearly independent eigenvectors associated with that eigenvalue is called its *geometric multiplicity* (GM) and satisfies $1 \leq \mathrm{GM} \leq \mathrm{AM}$. When GM = AM, there exists a full set of independent eigenvectors and $\mathbf{M}$ can be diagonalized. If GM < AM, no such set of eigenvectors exists and $\mathbf{M}$ is termed defective. Even if $\mathbf{M}$ is defective, it can always be brought to *Jordan normal form*, which can be thought of as almost diagonal. Specifically, there exists an invertible matrix $\mathbf{Q}$ such that $\mathbf{J} = \mathbf{Q}^{-1}\mathbf{M}\mathbf{Q}$ is block diagonal, with each block (called a Jordan block) containing repeated eigenvalues of $\mathbf{M}$ on its diagonal and 1s on its superdiagonal. Each eigenvector of $\mathbf{M}$ can be associated with a Jordan block, meaning the number of Jordan blocks equals the geometric multiplicity.

Suppose, for example, that $\mathbf{M} \in \mathbb{C}^{5\times 5}$ has a five-fold repeated eigenvalue λ with AM = 5 and GM = 2. In this case, $\mathbf{M}$ has two linearly independent eigenvectors, so $\mathbf{J}$ is a 5×5 matrix with two Jordan blocks. The sizes of the blocks must sum to 5, so the possible configurations are $2 + 3$ or $1 + 4$. These correspond to Jordan normal forms of the form

$$\left(\begin{array}{cc|ccc} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right), \qquad \left(\begin{array}{c|cccc} \lambda & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{array}\right). \tag{S1}$$

Which of these forms actually occurs is not determined by the GM alone, but depends on the precise structure of $\mathbf{M}$. In the second example above, the first Jordan block has size 1, so it has no superdiagonal. In fact, when GM = AM, all Jordan blocks are of size 1, meaning $\mathbf{J}$ is a diagonal matrix. In this case, $\mathbf{M}$ is not defective and the Jordan normal form is equivalent to the usual diagonal form $\mathbf{D}$.

2. PROOF OF DETERMINANT IDENTITY

Here we present a short proof of Eq. (4), i.e.,

$$\det(\omega \mathbf{I}_N - \mathbf{H}(\alpha)) = \det(\mathbf{S}^{-1}(\omega, \alpha)) \det(\omega \mathbf{I}_N - \mathbf{H}(\alpha) - i\mathbf{W}\mathbf{W}^\dagger). \tag{S2}$$

from the main text.

Consider a matrix $\mathbf{M}$ with block structure

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{S3}$$

where $\mathbf{A} \in \mathbb{C}^{N\times N}$, $\mathbf{B} \in \mathbb{C}^{N\times M}$, $\mathbf{C} \in \mathbb{C}^{M\times N}$, and $\mathbf{D} \in \mathbb{C}^{M\times M}$. If $\mathbf{A}$ and $\mathbf{D}$ are invertible then it is well known that [2]

$$\det(\mathbf{M}) = \det(\mathbf{A})\det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}). \tag{S4}$$

Setting $\mathbf{A} = \mathbf{I}_N$, $\mathbf{B} = \mathbf{RW}$, $\mathbf{C} = i\mathbf{W}^\dagger$, and $\mathbf{D} = \mathbf{I}_M$, where $\mathbf{R} = (\omega\mathbf{I}_N - \mathbf{H}(\alpha))^{-1}$, we find

$$\det(\mathbf{I}_M - i\mathbf{RWW}^\dagger) = \det(\mathbf{I}_N - i\mathbf{W}^\dagger\mathbf{RW}) = \det(\mathbf{S}), \tag{S5}$$

where the final equality follows from the definition of $\mathbf{S}$. Note now that

$$\det(\mathbf{I}_M - i\mathbf{RWW}^\dagger) = \det(\mathbf{R})\det(\mathbf{R}^{-1} - i\mathbf{WW}^\dagger) = \frac{\det(\omega\mathbf{I}_N - \mathbf{H}(\alpha) - i\mathbf{WW}^\dagger)}{\det(\omega\mathbf{I}_N - \mathbf{H}(\alpha))}. \tag{S6}$$

Substituting Eq. (S6) back into Eq. (S5) and rearranging yields Eq. (S2).

3. PROOF OF VANISHING LIMIT

In this section, we justify the limit

$$\lim_{\omega\to\omega_p} (\omega - \omega_p)^N \left(\frac{1}{g}\frac{\partial g}{\partial\alpha}\right)\bigg|_{\alpha=\alpha_0} = 0, \tag{S7}$$

from the main text where $g = \det(\omega\mathbf{I}_N - \mathbf{H}(\alpha) - i\mathbf{WW}^\dagger)$.

Recall first that $\mathbf{H}(\alpha_0)$ has the repeated eigenvalue ω_p. Note next that $\mathbf{WW}^\dagger$ is, by construction, Hermitian and positive semi-definite. Provided that $\mathbf{W}$ is not the zero matrix, $\mathbf{WW}^\dagger$ must possess at least one non-zero eigenvalue, implying $\mathrm{tr}(\mathbf{WW}^\dagger) \neq 0$. It follows that

$$\mathrm{tr}(\omega\mathbf{I}_N - \mathbf{H}(\alpha_0) - i\mathbf{WW}^\dagger) = N(\omega - \omega_p) - i\,\mathrm{tr}(\mathbf{WW}^\dagger) \tag{S8}$$

is non-zero at $\omega = \omega_p$. Since the trace of a matrix is equal to the sum of its eigenvalues, it hence follows that $\omega_p\mathbf{I}_N - \mathbf{H}(\alpha_0) - i\mathbf{WW}^\dagger$ possesses at least one non-zero eigenvalue. In other words, the addition of $-i\mathbf{WW}^\dagger$ necessarily shifts at least one of the eigenvalues of $\mathbf{H}(\alpha_0)$ away from ω_p. Consequently, factorization of $g(\omega, \alpha_0)$ will contain the factor $(\omega - \omega_p)$ with multiplicity strictly less than N. We can therefore write $g(\omega, \alpha_0) = (\omega - \omega_p)^{N-k}h(\omega)$ for some integer $1 \leq k \leq N$ where h is a polynomial necessarily satisfying $h(\omega_p) \neq 0$. Since $\partial g/\partial\alpha$ is finite at ω_p, the original limit can therefore be easily evaluated as

$$\lim_{\omega\to\omega_p} (\omega - \omega_p)^N \left(\frac{1}{g}\frac{\partial g}{\partial\alpha}\right)\bigg|_{\alpha=\alpha_0} = \lim_{\omega\to\omega_p} (\omega - \omega_p)^k \frac{1}{h}\frac{\partial g}{\partial\alpha}\bigg|_{\alpha=\alpha_0} = 0, \tag{S9}$$

as required.

4. EXAMPLE OF NO POLE MIXING AT A NON-TRIVIAL DIABOLIC POINT

In this section we present an example where the eigenvalues of $\mathbf{H}$ do *not* mix among the eigenvalues of $\mathbf{S}$, even though $\mathbf{WW}^\dagger$ is not diagonal. For this section we write $\mathbf{S} = \mathbf{I} - i\,\mathbf{W}^\dagger\mathbf{RW}$, $\mathbf{R} = \mathbf{QPQ}^{-1}$, and $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_N)$, where the diagonal entries $p_i = 1/(\omega - \omega_{p,i})$ contain the poles and $\mathbf{Q}$ diagonalizes $\mathbf{H}(\alpha_0)$. Note that although the eigenfrequencies $\omega_{p,i}$ coincide at $\alpha = \alpha_0$, their derivatives need not.

Take for example

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{S10}$$

A direct calculation gives

$$\mathbf{S} = \mathbf{I} - i \begin{pmatrix} 2p_2 & 2p_1 - 2p_2 & 2p_2 - 2p_3 & 2p_2 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & 2p_3 & 0 \\ 2p_2 & 2p_1 - 2p_2 & 2p_2 - 2p_3 & 2p_2 \end{pmatrix}, \tag{S11}$$

which has eigenvalues $1, 1 - ip_1, 1 - 4ip_2, 1 - 3ip_3$. Evidently, each eigenvalue of $\mathbf{S}$ that contains a pole depends on exactly one eigenvalue of $\mathbf{H}$, so there is no mixing of the eigenvalues of $\mathbf{H}$ into different eigenvalues of $\mathbf{S}$. This example, however, differs to the case presented in the main text since

$$\mathbf{W}\mathbf{W}^{\dagger} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{S12}$$

which is neither diagonal nor has equal diagonal entries. We stress that this example is not particularly special and many similar constructions can be produced.

REFERENCES

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